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# Model for mixed region collapse in a stratified fluid

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SUMMARY
The collapse of a mixed fluid mass immersed in a vertically stratified fluid is studied analytically. Simple approximate theories are given for the initial stage of the process for a fully mixed fluid mass in two and three dimensions and for a partially mixed fluid mass in two dimensions. The solutions are obtained via an energy conservation principle and they are exact within the models. They compare favorably with approximate solutions of the same model obtained previously for the two-dimensional fully mixed case.

## 1. Introduction

The gravitationally induced collapse of a well mixed fluid mass in an incompressible fluid having a stable vertical density gradient is a phenomenon which has attracted some attention in recent years [1–11]. The basic system is that of a static, cylindrically shaped homogeneous fluid mass embedded within a fluid having stable vertical density stratification. The homogeneous, or mixed, region is considered to be infinitely long in the axial direction so that the phenomenon is two-dimensional in a plane perpendicular to the cylinder axis. If the density of the mixed region is equal to that of the surrounding fluid at the level of the cylinder axis, the region will tend to flatten out or collapse into a sheet at the level of the axis since the fluid in the upper part of the mixed region is heavier, and that in the lower part lighter, than the surrounding fluid. Associated with this collapse process is radiation of energy from the collapsing mixed region into the surrounding fluid in the form of an internal gravity wave field. Properties of this system have been employed by various authors to model such physical phenomena as the turbulent wake of moving obstacles [5, 8–11], the flattening of airplane contrails [12], and the fine structure in the vertical density and velocity gradients in the oceans and atmosphere [13].

The purpose of this paper is to describe some very simple analytical models of the collapse process that are appropriate for the initial stage of motion. Two complementary formulations predict behavior that is somewhat different from the experiments of Wu [7] but is verified by the numerical solutions of Wessel [6]. A solution of a boundary value problem which is valid for asymptotically small time shows that the mixed region deforms into the shape of an ellipse. The second formulation uses this information and the fact that a hydrostatic pressure approximation at the mixed region boundary is equivalent to conservation of energy in the mixed region.

The solution for the two-dimensional cylindrical fully mixed case is compared with approximate solutions of the boundary value problem obtained previously by Mei [5] and Padmanabhan *et al.* [8]. The method also is utilized to obtain solutions for a two-dimensional region in which the fluid in the mixed portion is linearly stratified but is less stable than its surroundings. The partially mixed region exhibits oscillatory behavior for certain values of the stability ratio (the ratio of the fluid stability interior to the mixed region to the stability of the fluid exterior to the region). Finally, the method is utilized to obtain the solution for the axisymmetric collapse of a fully mixed spherical fluid mass immersed in the stratified fluid.

## 2. The mixed region collapse problem

The idealized physical problem, as shown in Figure 1, is to determine the flow phenomena when a circular region of radius $a_0$ of homogeneous fluid having density $\rho_0$ is released from rest.

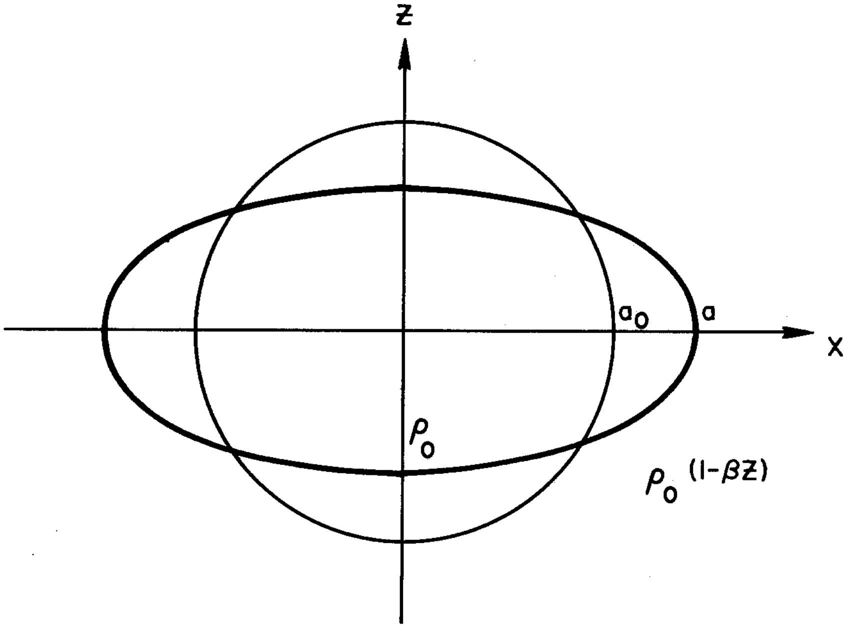


Figure 1. Coordinate system.

This mixed region is surrounded by a stratified fluid of density $\rho(z)=\rho_0(1-\beta z)$ of infinite extent. The constant $\beta$ is an inverse length scale of the stratification and the quantity

$$N = \left(-\frac{g}{\rho}\frac{d\rho}{dz}\right)^{\frac{1}{2}} = (\beta g)^{\frac{1}{2}}$$

is called the Väisälä–Brunt frequency of the stratified fluid.

In all real applications, one expects that the fluid viscosity and the diffusivity of whatever material (or heat) that causes the density stratification would play a role in the resulting fluid motions. This is true here but significant effects are expected only at a later stage in the collapse. To illustrate this, consider a Reynolds number that may be defined for the collapsing region. A length scale is the radius of the mixed region, $a_0$, and a velocity scale is $Na_0$. Thus, a Reynolds number may be defined as

$$\mathrm{Re} = \frac{Na_0^2}{\nu}$$

where, for the experiment of Wu [7], this number is the order of $10^4$. It would be bigger for larger-scale applications. When the mixed region deforms, the height becomes less so that a Reynolds number based upon this length scale would become smaller with increasing time. Thus, though viscous or diffusive effects will ultimately become important, an estimate of the earliest time of importance might be half a dozen Väisälä–Brunt periods. Consequently, in the initial stage, these effects will be neglected.

The collapse process is characterized by the dynamic response of the mixed region and its environment to the buoyancy force arising from the action of gravity on the perturbation of the environmental density field which defines the mixed region. At this time, no complete analytical description of the collapse process exists. However, several authors [5, 8] have suggested a model problem approach in which the dynamic response is restricted to the fluid within the mixed region. Although the motions in the surrounding fluid may be comparable in magnitude to those within the region, it is expected that this simplified model retains enough of the essential physics to provide a viable qualitative description of the initial accelerative stage of the collapse process. The essential simplification introduced by the assumption that motions are restricted to the fluid within the mixed region is that the pressure at the boundary of the mixed region can be taken to be the hydrostatic pressure there. In approaching this model, a boundary value problem can be set up for the resulting fluid motions in the homogeneous region. Defining the velocity potential $\varphi$ by the relation $\boldsymbol{v} = -\nabla\varphi$, the boundary value problem is

$$\begin{aligned} &\nabla^2\varphi = 0\,, && \text{in mixed region} \\ &\frac{\partial\varphi}{\partial t} - \tfrac{1}{2}q^2 = \frac{N^2 z^2}{2}\,, && \text{on } B \end{aligned} \tag{1}$$

where $q$ is the fluid speed. Using polar coordinates defined by the relations $x = r\cos\theta$, $z = r\sin\theta$, the velocity potential may be assumed to be of the form

$$\varphi(r,\theta,t) = \sum_{n=0}^{\infty} b_n(t)\, r^n \exp(in\theta)\,. \tag{2}$$

This satisfies the governing equation exactly and the coefficients are obtained from the pressure boundary condition so that, to lowest order,

$$\varphi(r,\theta,t) = \frac{N^2 t}{4}\,(a_0^2 - r^2\cos 2\theta)\,. \tag{3}$$

The resulting velocity field corresponds to stagnation point flow which indicates that the circle deforms into the shape of an ellipse whose major axis is horizontal. The half-width of the mixed region is given by the expression

$$a(t) = a_0\left\{1 + \frac{N^2 t^2}{4} - O(N^4 t^4)\right\}. \tag{4}$$

This solution could be extended to longer times by several techniques including the series method used by Penney and Thornhill [14] but the method presented in the following section seems more fruitful.

### 3. Similarity solution

The mathematical model (1) assumes continuity of pressure across the mixed region interface and that the pressure at the interface is the local hydrostatic pressure. This implies that no work can be done on the surrounding fluid by the mixed region and, consequently, that the total energy of the mixed region is conserved.

In order to determine the energies of the mixed region, a similarity solution is assumed in which the shape of the region at any time may be obtained by a linear affine transformation of the shape at any other time. That is, the initially circular region will deform in the shape of an ellipse. The flow field inside the region which will produce a simple stretching transformation of its shape is stagnation point flow,

$$\boldsymbol{v}\cdot\boldsymbol{i}_1 = x f(t) \tag{5a}$$

$$\boldsymbol{v}\cdot\boldsymbol{i}_2 = -z f(t)\,. \tag{5b}$$

It was shown in the previous section that this situation does obtain in the initial stage where that solution is valid. Now, the unknown function $f(t)$ may be replaced by the expression

$$f(t) = \frac{1}{a(t)}\frac{da}{dt} \tag{6}$$

where $a(t)$ is the half-width of the mixed region.

The kinetic energy is given by

$$K = \frac{\rho_0}{2}\iint |v|^2\, dx\, dz$$

where the integration extends over the entire mixed region. For the stagnation point flow (5a–b) with expression (6), the kinetic energy evaluates to

$$K = \rho_0 \frac{\pi}{8} \left(\frac{da}{dt}\right)^2 a_0^2 (1 + a_0^4 a^{-4}) \tag{7}$$

where $\pi a_0^2$ is the area of the region. The potential energy is determined as the work expended against the buoyancy force in creating the mixed region as a perturbation of the background, so that

$$P = g \iint \left\{ \int_0^z [\rho_0 - \rho(z')] dz' \right\} dx dz$$

which evaluates to

$$P = \rho_0 \frac{\pi}{8} N^2 a_0^6 a^{-2} . \tag{8}$$

Introducing the nondimensional variables

$$\begin{aligned} \alpha &= a/a_0 \\ \tau &= Nt , \end{aligned} \tag{9}$$

the conservation of total energy

$$K + P = \text{constant}$$

implies that

$$(1+\alpha^{-4})\dot{\alpha}^2 + \alpha^{-2} - 1 = 0 . \tag{10}$$

The constant is determined from the potential energy available in the mixed region when initially at rest. Considering $\tau$ as the dependent variable, equation (10) may be integrated directly, yielding the solution

$$\tau = \int_1^\alpha \left\{ \frac{1+\zeta^{-4}}{1-\zeta^{-2}} \right\}^{\frac{1}{2}} d\zeta . \tag{11}$$

It is readily verified that this represents an exact solution to the problem posed by Mei [5] and Padmanabhan *et al.* [8].

The solution (11) is plotted in Figure 2, together with the results of the analyses by Mei [5] and Padmanabhan *et al.* [8]. Mei [5] obtained a solution that is somewhat simpler than that given by expression (11). In particular, his solution neglects the kinetic energy of vertical motions and, correspondingly, his solution outdistances the solution (11). The numerical results of Padmanabhan *et al.* [8] are inexplicable. Since viscous effects are negligible for such small times, their inviscid and viscous solutions should be practically identical. There is no obvious reason why their solutions do not agree with each other and do not compare well with the present results.

## 4. Partially mixed and three-dimensional regions

The techniques developed in the preceding section are readily extended to partially mixed and three-dimensional regions. As a model for partial mixing, consider the system described in section 2, with the density within the region given by

$$\rho_i(z) = \rho_0 (1 - \beta_i z)$$

where $\beta_i$ must be considered as a function of time. Assuming that the collapse process commences from a state of rest, then, since the density within the region is independent of $x$, the vorticity in the region is essentially zero for such times as viscous effects are negligible, and the stagnation point flow (5a, b) is a correct solution for the internal velocity field. Introducing the

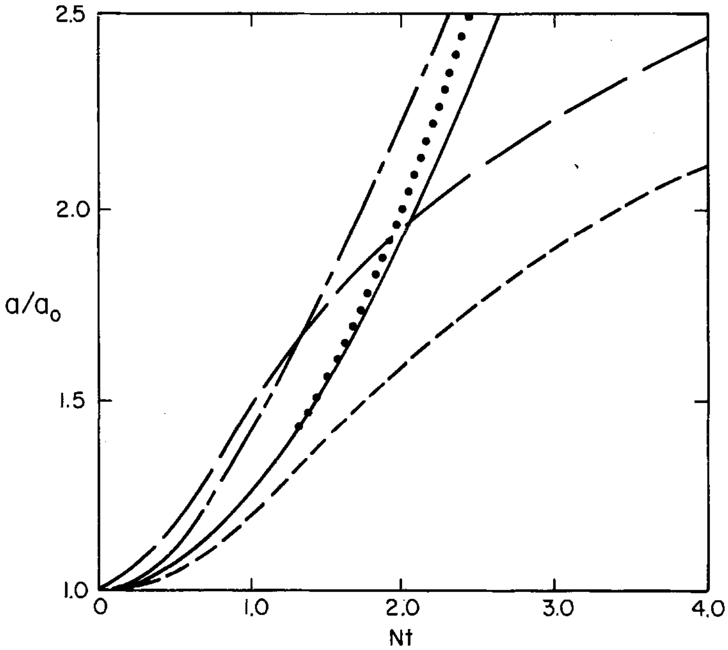


Figure 2. Variation of mixed region half-width *versus* time for two-dimensional fully mixed region. ......... asymptotic result, equation (4), ——— exact result, equation (11), — — — approximate result, Mei [5], ——— inviscid numerical result, Padmanabhan *et al.* [8], ------- viscous numerical result, Padmanabhan *et al.* [8].

mixing parameter

$$\varepsilon = \beta_i(0)/\beta\,, \tag{12}$$

which is zero for the fully mixed case treated previously and approaches unity for very small mixing, the potential energy in the partially mixed region is given by

$$P = g\iint \left\{ \int_{z-\delta}^{z} [\rho_i(z)-\rho(z')]\,dz' \right\} dx\,dz$$

where $\delta$ is the displacement of a fluid particle from its equilibrium position. This displacement is given by

$$\delta = (1-\beta_i/\beta)\,z$$

and, hence, the potential energy is

$$P = \rho_0 \frac{\pi}{8} N^2 (1-\varepsilon a/a_0)^2 a_0^6 a^{-2}\,. \tag{13}$$

The kinetic energy is changed from expression (7) by a term of order $\beta_i a_0 \ll 1$. However, this term is of negligible importance in practical situations and it is omitted in this analysis. The omission of this term is not necessary but it is consistent with the Boussinesq approximation (*cf.* [15]). The conservation principle yields the differential equation

$$(1+\alpha^{-4})\dot{\alpha}^2 + (1-2\varepsilon\alpha)\alpha^{-2} - (1-2\varepsilon) = 0 \tag{14}$$

for the variation of the mixed region half-width with time.

An interesting feature of equation (14) is that its solution is oscillatory for $\varepsilon > \frac{1}{2}$. That is, for sufficiently small mixing, the region undergoes a nonlinear oscillation. In the limit $\varepsilon \to 1$, the motion becomes sinusoidal with nondimensional frequency $2^{-\frac{1}{2}}$. The frequency of oscillation has been evaluated numerically and is plotted in Figure 3 as a function of the mixing parameter $\varepsilon$. It is a characteristic of the idealized model considered here that the oscillations are undamped.

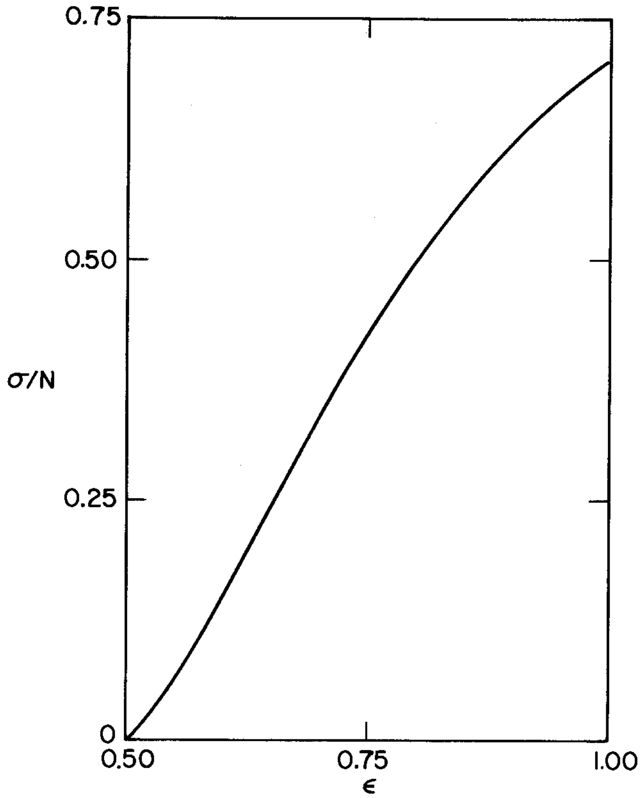


Figure 3. Nondimensional frequency of oscillation, $\sigma/N$, *versus* mixing parameter, $\varepsilon$, for partially mixed two-dimensional region.

If a more realistic model including energy transfer to the surrounding medium were constructed, any oscillations of the region would, of course, be damped as energy is fed into the internal wave field in the surrounding medium. Hartman and Lewis [10] have illustrated this in the limiting case of $\varepsilon \to 1$, for which the linear theory becomes valid.

The three-dimensional generalization of the mixed region problem, that is, the collapse of an initially spherical mixed region, may also be considered in this framework. The appropriate similarity shape is an oblate spheroid, the interior velocity field being given by

$$\boldsymbol{v} \cdot \boldsymbol{i}_1 = xf(t) \tag{15a}$$

$$\boldsymbol{v} \cdot \boldsymbol{i}_2 = yf(t) \tag{15b}$$

$$\boldsymbol{v} \cdot \boldsymbol{i}_3 = -2zf(t) . \tag{15c}$$

The kinetic and potential energies are obtained as integrals over an ellipsoidal volume rather than over an ellipse as before so that

$$K = \frac{4\pi}{15} \rho_0 a^{-2} a_0^5 \left(\frac{da}{dt}\right)^2 (a^2 a_0^{-2} + 2a^{-4} a_0^4) \tag{16a}$$

$$P = \frac{2\pi}{15} \rho_0 N^2 a^{-4} a_0^9 . \tag{16b}$$

With the definitions (6) and (9), the conservation principle yields the equation

$$(1+2\alpha^{-6})\dot{\alpha}^2 + \tfrac{1}{2}\alpha^{-4} - \tfrac{1}{2} = 0 . \tag{17}$$

This may be integrated directly to yield the solution

$$\tau = 2^{\frac{1}{2}} \int_1^{\alpha} \left\{ \frac{1+2\zeta^{-6}}{1-\zeta^{-4}} \right\}^{\frac{1}{2}} d\zeta \, . \tag{18}$$

This is the solution of the axisymmetric problem and it is plotted in Figure 4. This solution, like that in the two-dimensional case, exhibits an initial quadratic dependence on time and a later linear dependence on time. The initial dependence on time is given by

$$a(t) = a_0 \left\{ 1 + \frac{N^2 t^2}{6} - O(N^4 t^4) \right\} \tag{19}$$

and this agrees with the asymptotic solution of the three-dimensional boundary value problem as given by equations (1). In the three-dimensional case, the general solution that corresponds to expression (2) involves Legendre polynomials and, although tedious, the derivation of expression (19) from that solution is straightforward.

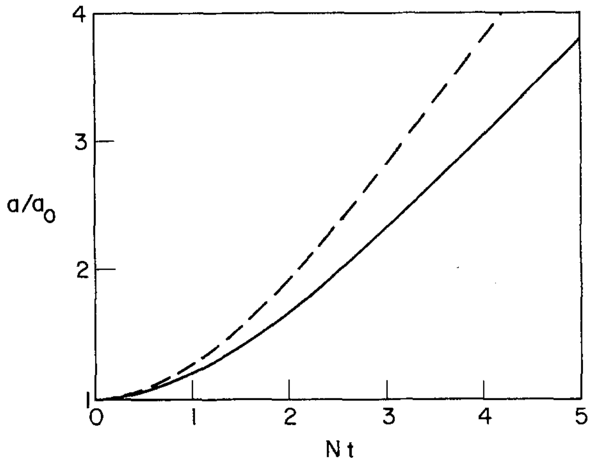


Figure 4. Variation of mixed region half-width *versus* time, ——— three-dimensional result, equation (18), – – – two-dimensional result, equation (11).

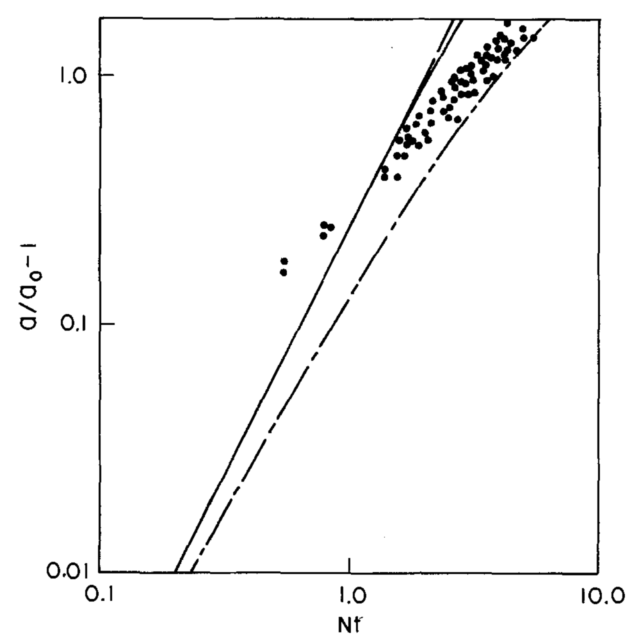


Figure 5. Variation of mixed region half-width versus time for two-dimensional fully mixed region, – – – asymptotic result, equation (4), ——— exact result, equation (11), — — — numerical result, Wessel [6], ......... experimental data, Wu [16].

## 5. Conclusions

The primary results are given in Figures 3–5. The fully mixed two-dimensional case is the only one that has received much attention in the past. Mei [5] and Padmanabhan *et al.* [8] obtained approximate solutions of models of the collapse process that are identical to the model solved in closed form here. The comparison with their solutions is given in Figure 3. Also, Wu [7, 16] has obtained experimental results for the physical problem modelled here. The results are shown in Figure 5 along with the analytical results obtained here. The results of Wu [7, 16] do not show a quadratic time dependence for small time as do the present results. In fact, Wu [7] assumed that the mixed region width in the initial stage increases practically linearly with time. Mei [5] has pointed out that the early time width of the mixed region physically cannot increase faster than quadratically with time. Mei's [5] conclusions are substantiated by the present results but, anyway, the early time dependence might have been masked by the experimental method. The corresponding result predicted by Wessel [6] by a numerical solution of the Navier–Stokes equations is also shown in Figure 5 and the time variation in the initial stage does agree well with our theoretical results. The shape of the mixed region used by Wessel was square instead of round but, for such times as the initially square region can be approximated by a rectangle, and motion in the surrounding fluid neglected, a similarity approach like that employed in this paper yields exactly the same result as obtained here.

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